Algebra-II
B. Math - First year
Mid-Semestral Exam
2012-2013

Time: 3hrs
Max score: 100

Answer question **1** and any **four** from the rest.

(1) State true or false. Justify your answers. No marks will be awarded in absence of proper justification.
(a) 0 is not an eigenvalue of the differential operator $D : \mathcal{P}_n(\mathbb{R}) \longrightarrow \mathcal{P}_n(\mathbb{R})$, where $\mathcal{P}_n(\mathbb{R})$ is the space of all real polynomials of degree less than or equal to n.
(b) There exist invertible skew symmetric real matrices of odd order.
(c) For a field $\mathbb{F}$, bases of the vector space $\mathbb{F}^n$ are in bijective correspondence with elements of $GL_n(\mathbb{F})$.
(d) If A is an $n \times n$ matrix such that $A^2 = A$, then rank (A) + rank $(I_n - A) = n$. 8+8+8+8

(2) (a) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & 2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}.$$

(b) Let A be the $n \times n$ backward identity matrix= $\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$.
What is determinant of A? Find A^{-1}. 8+9

(3) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z)^t = (x+y+2z, x-y-3z, 2x+3y+z)^t$. Let $\mathcal{B}_1$ be the standard basis and $\mathcal{B}_2 = ((1, 1, 1)^t, (1, -1, 1)^t, (1, 1, 2)^t)$ be another ordered basis of $\mathbb{R}^3$. Then find
(a) the matrix of T with respect to $\mathcal{B}_1$, say A_1,
(b) the matrix of T with respect to $\mathcal{B}_2$, say A_2,
(c) a matrix P such that $PA_1P^{-1} = A_2$. 5+6+6

Please turn over

(4) (a) Show that A and A^t have the same set of eigenvalues. Give examples to show that eigen vectors of A and A^t may be different.
(b) Let $A = (a_{ij})$ be an $n \times n$ matrix. Suppose that for all $i, 1 \leq i \leq n$, $\sum_{j=1}^{n} a_{ij} = 1$. Then prove that 1 is an eigenvalue of A. What is the corresponding eigenvector?
(c) Suppose all the column sums of A equal to 1. Does the same result hold? 5+8+4

(5) (a) Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be a linear operator such that $\text{rank}(T - I) = 3$ and null-space(T)= $\{(x_1, x_2, x_3, x_4, x_5)^t \in \mathbb{R}^5 : x_1 + x_4 + x_5 = 0,\ x_2 + x_3 = 0\}$.
(a) Determine the eigenvalues of T ?
(b) Find the number of linearly independent eigenvectors corresponding to each eigenvalue?
(c) Is T diagonalizable? Justify your answer. 6+7+4

(6) (a) Define row rank, column rank of a matrix.
(b) Show that row rank of a matrix is equal to its column rank.
(b) Let A be a $m \times n$ matrix and B be a $n \times k$ matrix. Prove that rank $(AB) \leq$ min $\{\text{rank}(A), \text{rank}(B)\}$. 2+7+8